Mathematics I

Land Resources Managment

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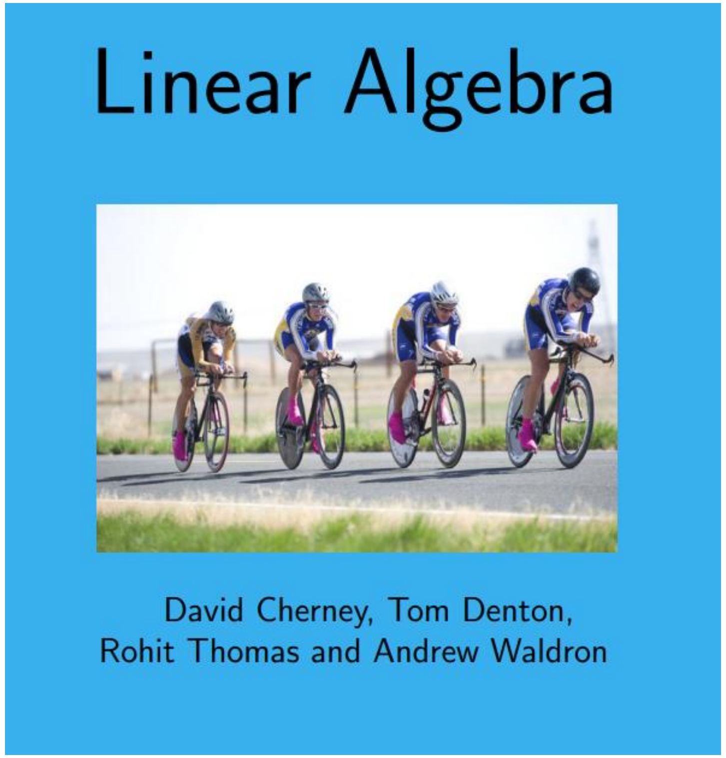
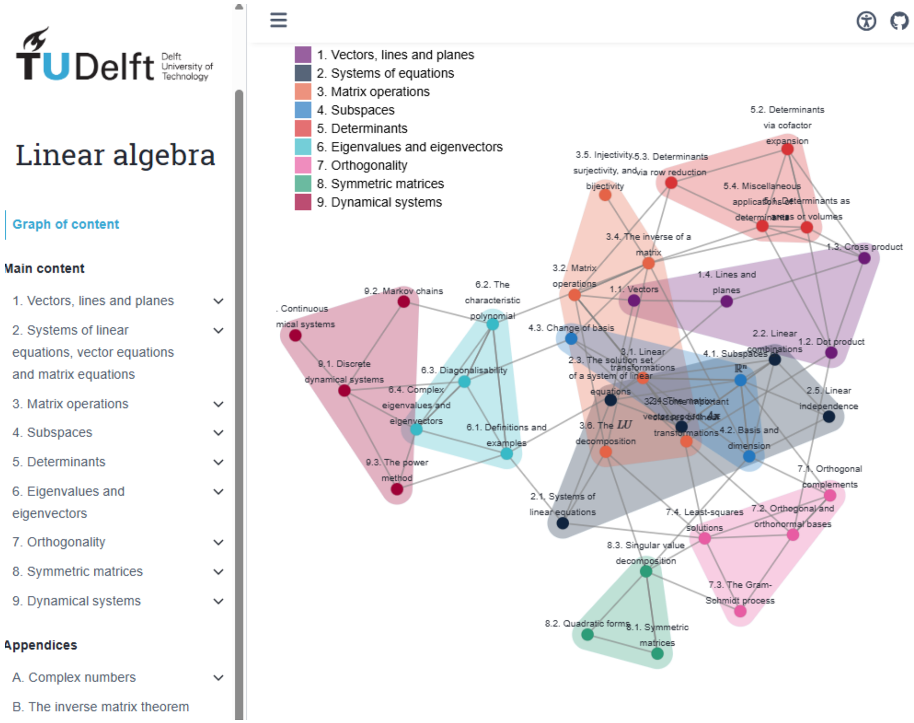
Course Completion Requirements:

1. **Attendance:**
   Regular attendance at all classes is required.
2. **Active participation:**
   Students are expected to actively engage in problem-solving activities during tutorials.
3. **Assessment of tutorials:**
   There will be **four midterm tests**, held after every two tutorials.
   To pass the tutorial component, students must obtain **at least 50% of the total points** available from all four tests combined.
   Each test carries the same number of points.
4. **Make-up test policy:**
   Failure to achieve the required total score results in a **comprehensive make-up test** covering all the material from the tutorials.
5. **Final grade:**
   The final course grade is based on:
   • passing the tutorial component, and
   • the grade for the **presentation**.
   The final grade is the **arithmetic mean** of these two components (tutorial + presentation).
   Presentation topics will be assigned **by mid-December**.
   A student who disagrees with the final grade may take a **comprehensive final exam** covering both theoretical and problem-solving parts.

It's worth taking a look

1. https://www.math.ucdavis.edu/~linear/linear-guest.pdf
2. https://interactivetextbooks.tudelft.nl/linear-algebra/index.html





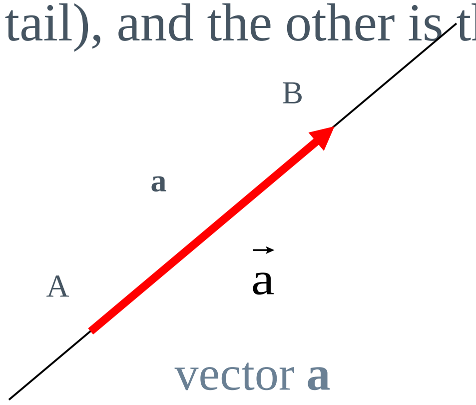
Course Program – Linear Algebra

1. **Vector spaces, vectors, and unit vectors (versors)**
2. Types of matrices, operations on matrices, matrix rank
3. Determinants and their properties
4. Inverse matrix
5. Systems of linear equations – Cramer's rule
6. Methods of solving systems of equations, Gaussian elimination method
7. Eigenvalues and eigenvectors
8. Generalized inverse matrix (optional / to be confirmed)

VECTOR SPACES
Vectors in R^n

Definition:

A *vector* is an ordered pair of points. One point is the *initial point* (the tail), and the other is the *terminal point* (the head).



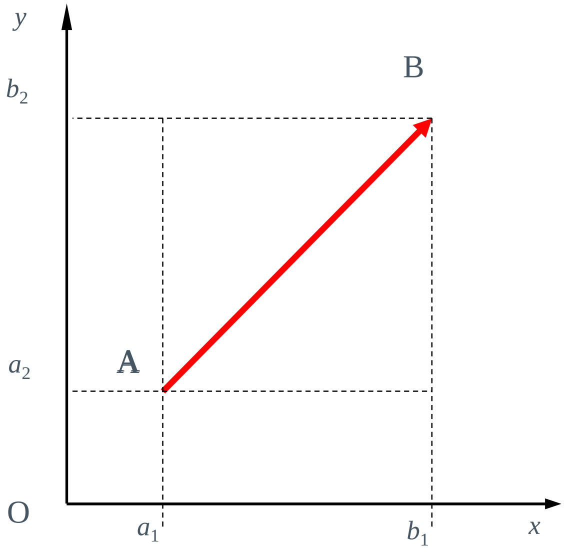
A vector possesses **direction**, **orientation (sense)**, and **magnitude**.

- The **direction** of a vector is the line on which it lies.
- The **orientation** (sense) specifies which of the two possible directions along that line is taken — from the initial to the terminal point.
- The **magnitude** (length) of a vector is the distance between its initial and terminal points, expressed in given units.

We denote a vector from point A to point B as $\overrightarrow{AB}$ or by lowercase letters such as a,b,c, or as $\vec{a}, \vec{b}, \vec{c}$.

The **magnitude** (length) of vector is denoted by $|\overrightarrow{AB}| = |a|$

Vector Space



- **Definition :**

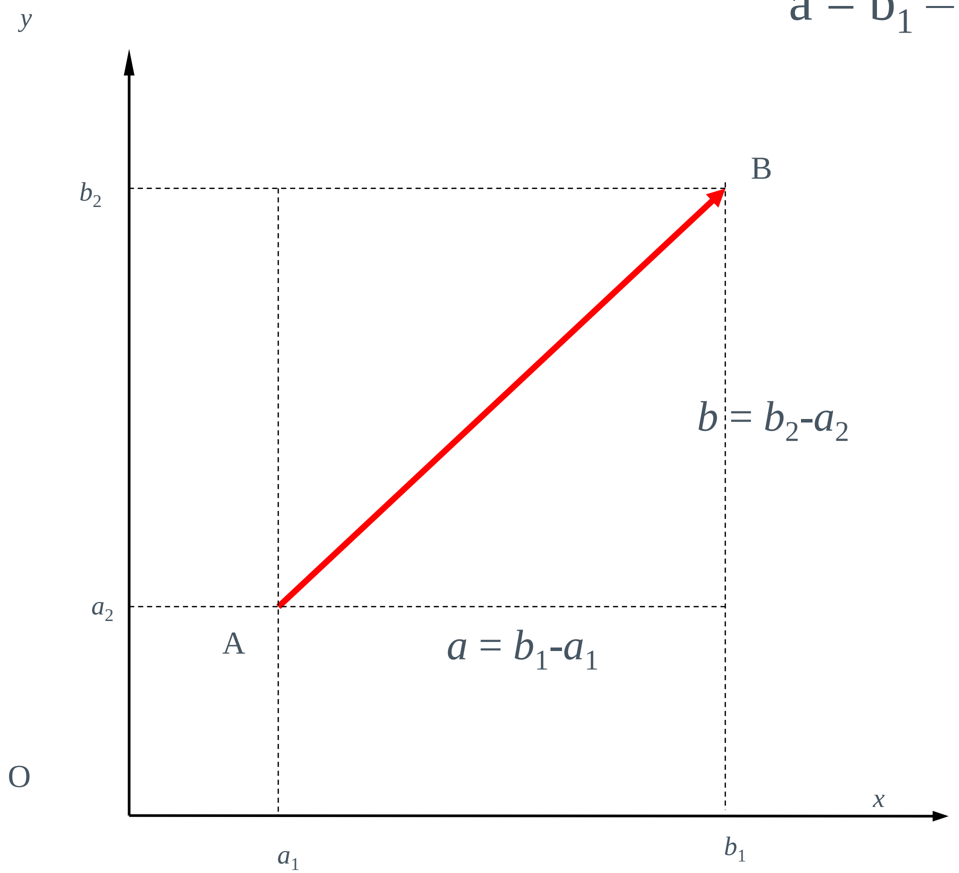
  The set R^n equipped with vector addition and scalar multiplication by real numbers is called an *n-dimensional real vector space*.

Vector Space

Components of a Vector – Space R^2

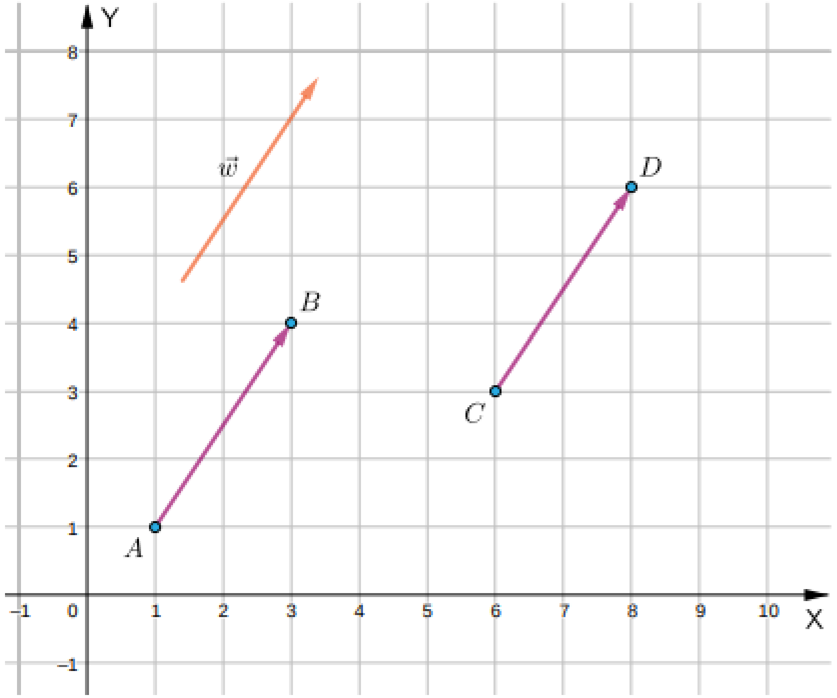
Let the point $A=(a_1, a_2)$ be the initial point of a vector and $B=(b_1, b_2)$ its terminal point. Then the *components* of the vector are determined by subtracting the corresponding initial coordinates from the terminal coordinates of the vector:

$$a = b_1 - a_1,\ b = b_2 - a_2.$$



$$\overrightarrow{AB} = \mathbf{u} = \begin{bmatrix} b_1 - a_1 \\ b_2 - a_2 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}.$$

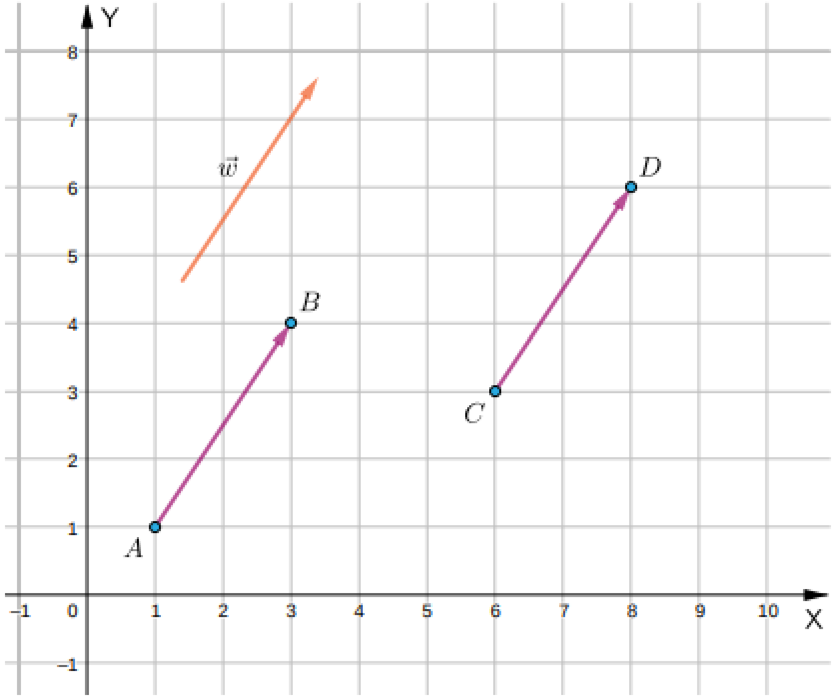
A **free vector** is the set of all directed segments that have the same length, direction, and orientation — that is, all vectors that are equal.



Since, in the case of free vectors, the notions of the initial point and the terminal point lose their significance, we usually denote free vectors by lowercase Latin letters: **a**, **b**, **c**, **v**, w, etc.

We can say that if two directed segments $\overrightarrow{AB}$ and $\overrightarrow{CD}$ have the same direction, orientation, and length, then they represent the same free vector.

A **bound vector** (or attached vector) is a free vector whose initial point is fixed at a specific point called the *point of application*.



Vector Space

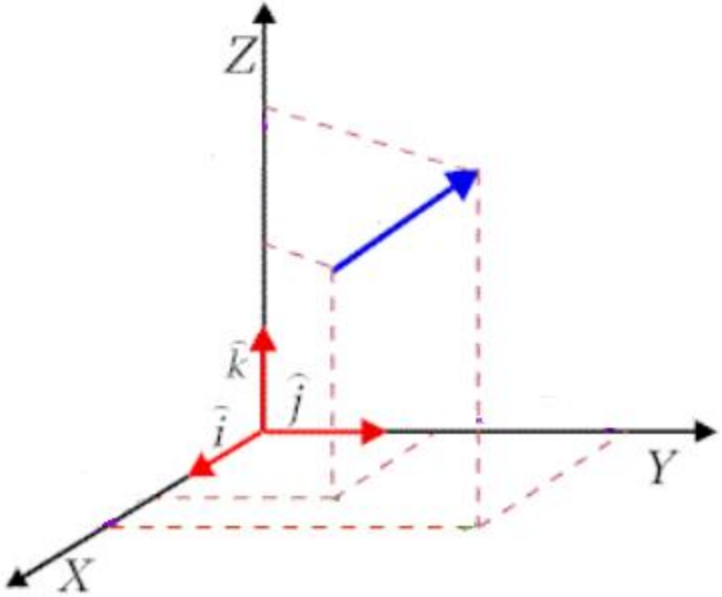
Components of a Vector – Space R^3

Let $A=(a_1, a_2, a_3)$ be the initial point of a vector and $B=(b_1, b_2, b_3)$ – its terminal point.

Then the *components* of the vector are given by:

$$a = b_1 - a_1,\ b = b_2 - a_2,\ c = b_3 - a_3.$$

$$\overrightarrow{AB} = \mathbf{u} = \begin{bmatrix} b_1 - a_1 \\ b_2 - a_2 \\ b_3 - a_3 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$



Vector Space

Components of a Vector – Space R^n

For A=$(a_1, a_2, \ldots, a_n)$ and B=$(b_1, b_2, \ldots, b_n)$

Then the *components* of the vector are determined by subtracting the corresponding initial coordinates from the terminal coordinates of the vector:

$$b_1 - a_1,\ b_2 - a_2,\ \ldots,\ b_n - a_n.$$

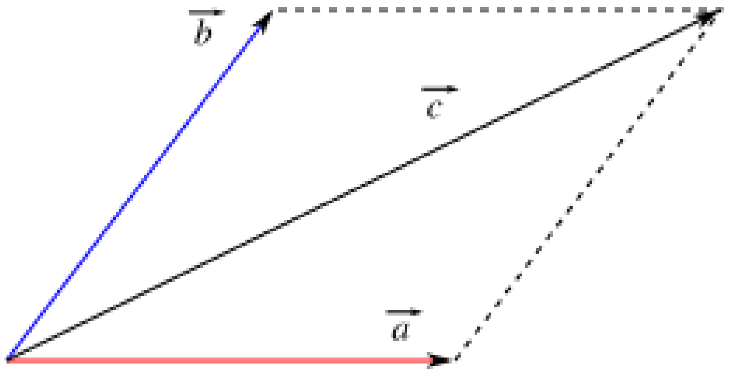
$$\overrightarrow{AB} = \mathbf{u} = \begin{bmatrix} b_1 - a_1 \\ b_2 - a_2 \\ \cdots\cdots\cdots \\ b_n - a_n \end{bmatrix}.$$

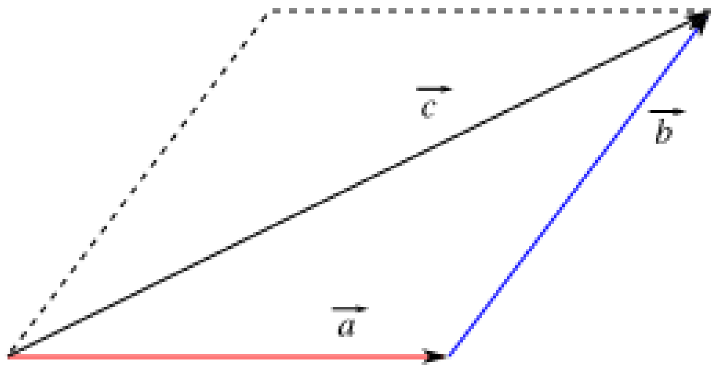
OPERATIONS ON FREE VECTORS

Addition of Vectors

Let $\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \in \mathbb{R}^n.$

The **sum** of vectors **a** and **b** is the vector **c** defined by $\mathbf{c} = \mathbf{a} + \mathbf{b} = \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \vdots \\ a_n + b_n \end{bmatrix}.$



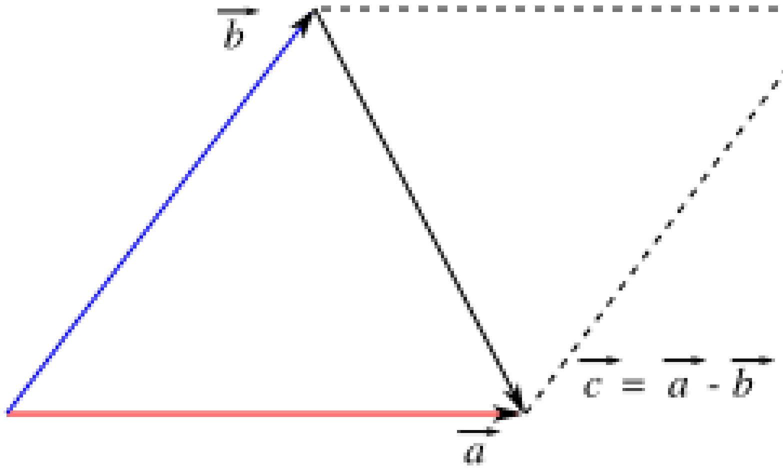


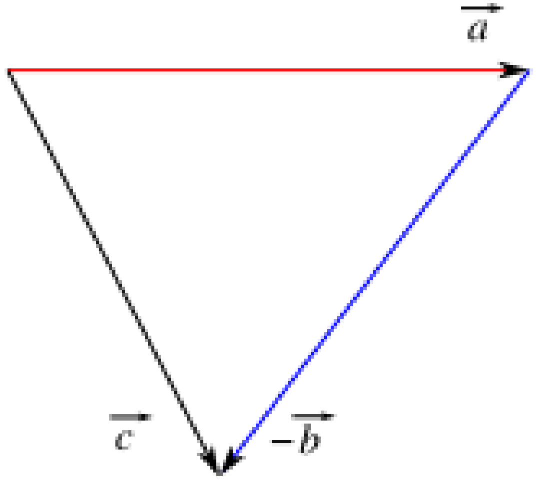
OPERATIONS ON FREE VECTORS

Subtraction of Vectors

The **difference** of vectors **a** and b is the vector **c** defined by

$$\mathbf{c} = \mathbf{a} - \mathbf{b} = \begin{bmatrix} a_1 - b_1 \\ a_2 - b_2 \\ \vdots \\ a_n - b_n \end{bmatrix}.$$





OPERATIONS ON FREE VECTORS

Scalar Multiplication

For a scalar $k \in \mathbb{R}$ and a vector

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix},$$

the **product** of k and $\mathbf{a}$ is the vector

$$k\mathbf{a} = \begin{bmatrix} ka_1 \\ ka_2 \\ \vdots \\ ka_n \end{bmatrix}.$$

OPERATIONS ON FREE VECTORS

Opposite and Zero Vectors

The **opposite vector** (or **negative**) of **a** is

$$-\mathbf{a} = \begin{bmatrix} -a_1 \\ -a_2 \\ \vdots \\ -a_n \end{bmatrix}.$$

The **zero vector** (or **null vector**) in R^n is

$$\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

OPERATIONS ON FREE VECTORS

Properties of Vector Operations

For all $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^n$ and $k, h \in \mathbb{R}$:

1. $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$ *(commutativity of addition)*
2. $\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$ *(associativity of addition)*
3. $\mathbf{a} + \mathbf{0} = \mathbf{0} + \mathbf{a} = \mathbf{a}$ *(existence of a zero element)*
4. $\mathbf{a} + (-\mathbf{a}) = (-\mathbf{a}) + \mathbf{a} = \mathbf{0}$ *(existence of additive inverse)*
5. $k(\mathbf{a} + \mathbf{b}) = k\mathbf{a} + k\mathbf{b}$ *(distributivity over vector addition)*
6. $(k + h)\mathbf{a} = k\mathbf{a} + h\mathbf{a}$ *(distributivity over scalar addition)*
7. $(kh)\mathbf{a} = k(h\mathbf{a})$ *(associativity of scalar multiplication)*
8. $1\mathbf{a} = \mathbf{a}$ *(multiplicative identity)*.

VECTOR SPACES
Linear Combination

Definition:

A **linear combination** of vectors

$$\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k \in \mathbb{R}^n$$

is any vector $\mathbf{b} \in \mathbb{R}^n$ that can be written as

$$\mathbf{b} = \lambda_1 \mathbf{a}_1 + \lambda_2 \mathbf{a}_2 + \ldots + \lambda_k \mathbf{a}_k,$$

where $\lambda_1, \lambda_2, \ldots, \lambda_k \in \mathbb{R}$ are real scalars.

Theorem:

Every linear combination of vectors in $\mathbb{R}^n$ is also a vector in $\mathbb{R}^n$.

VECTOR SPACES

Norm of a Vector

Definition:

The **norm** (or **length**) of a vector is defined as:

$$|\mathbf{a}| = \begin{cases} \sqrt{a_1^2 + a_2^2}, & \text{for } \mathbf{a} \in \mathbb{R}^2, \\ \sqrt{a_1^2 + a_2^2 + a_3^2}, & \text{for } \mathbf{a} \in \mathbb{R}^3, \\ \sqrt{a_1^2 + a_2^2 + \ldots + a_n^2}, & \text{for } \mathbf{a} \in \mathbb{R}^n. \end{cases}$$

Properties of the Vector Norm:

Let $\mathbf{a} \in \mathbb{R}^n$ and $k \in \mathbb{R}$. Then:

1. $|\mathbf{a}| \geq 0$, and $|\mathbf{a}| = 0 \iff \mathbf{a} = \mathbf{0}$.
2. $|k\mathbf{a}| = |k|\,|\mathbf{a}|$.

Dot Product (Scalar Product)

Definition:

The **dot product** (or **scalar product**) of two vectors **a** i **b** is defined as:

$$\mathbf{a} \circ \mathbf{b} = \begin{cases} a_1 b_1 + a_2 b_2, & \text{for } \mathbf{a}, \mathbf{b} \in \mathbb{R}^2, \\ a_1 b_1 + a_2 b_2 + a_3 b_3, & \text{for } \mathbf{a}, \mathbf{b} \in \mathbb{R}^3, \\ a_1 b_1 + a_2 b_2 + \ldots + a_n b_n, & \text{for } \mathbf{a}, \mathbf{b} \in \mathbb{R}^n. \end{cases}$$

Geometric Interpretation:

If α is the angle between two non-zero vectors $\mathbf{a}$ and $\mathbf{b}$, then

$$\mathbf{a} \circ \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \cos \alpha.$$

Properties of the Dot Product

Theorem

For all $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^n$ and $k \in \mathbb{R}$:

1. $\mathbf{a} \circ \mathbf{b} = \mathbf{b} \circ \mathbf{a}$ *(commutativity)*
2. $\mathbf{a} \circ (\mathbf{b} + \mathbf{c}) = \mathbf{a} \circ \mathbf{b} + \mathbf{a} \circ \mathbf{c}$ *(distributivity)*
3. $(k\mathbf{a}) \circ \mathbf{b} = k(\mathbf{a} \circ \mathbf{b}) = \mathbf{a} \circ (k\mathbf{b})$ *(compatibility with scalar multiplication)*
4. $\mathbf{a} \circ \mathbf{a} = |\mathbf{a}|^2$
5. $\mathbf{a} \circ \mathbf{0} = \mathbf{0} \circ \mathbf{a} = 0$

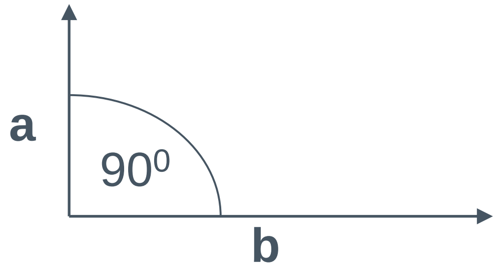
VECTOR SPACES

Orthogonal Vectors

Definition:

Two vectors $a, b \in \mathbb{R}^n$ are said to be **orthogonal** (perpendicular) if and only if

$$\mathbf{a} \circ \mathbf{b} = 0$$



Since :

$$\mathbf{a} \circ \mathbf{b} = |\mathbf{a}|\ |\mathbf{b}| \cos\alpha.$$

$$\cos 90^0 = 0$$

VECTOR SPACES

VECTOR TRANSPOSITION

COLUMN VECTOR

$$\mathbf{a} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

ROW VECTOR

$$\mathbf{b} = [d \quad e \quad f]$$

vector transposition

$$\mathbf{a}^{\mathrm{T}} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}^{\mathrm{T}} = [a \quad b \quad c]$$

$$\mathbf{b}^{\mathrm{T}} = [d \quad e \quad f]^{\mathrm{T}} = ?$$

UNIT VECTOR (VERSOR)

Definition:

A **unit vector** (or **versor**) is a vector whose length equals 1:

Theorem:

The unit vector **parallel** to a given non-zero vector **a** is defined as

$$\mathbf{e}_a = \frac{\mathbf{a}}{|\mathbf{a}|}.$$

VECTOR SPACES

Axial Unit Vectors:

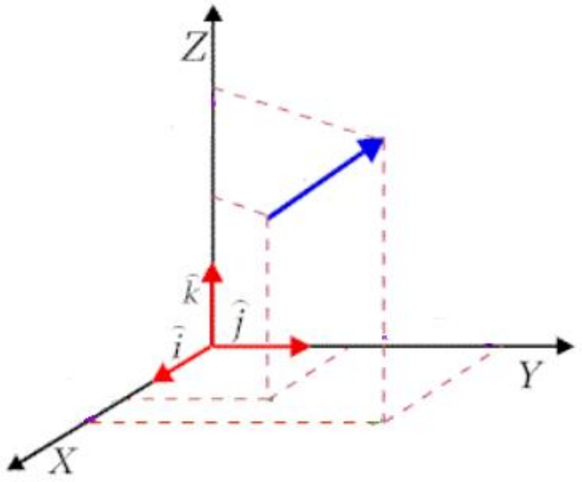
- In the two-dimensional space $\mathbb{R}^2$:

$$\mathbf{i} = [1, 0]^T, \quad \mathbf{j} = [0, 1]^T.$$

- In the three-dimensional space $\mathbb{R}^3$:

$$\mathbf{i} = [1, 0, 0]^T, \quad \mathbf{j} = [0, 1, 0]^T, \quad \mathbf{k} = [0, 0, 1]^T.$$

Each of these vectors has length 1 and direction corresponding to one of the positive coordinate axes of a Cartesian coordinate system.



Theorem:

Every non-zero vector can be expressed as a linear combination of unit vectors along the coordinate axes (called axial unit vectors).