Mathematics I

Land Resources Management

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Course Program – Linear Algebra

1. Vector spaces, vectors, and unit vectors (versors)
2. **Types of matrices, operations on matrices, matrix rank**
3. Determinants and their properties
4. Inverse matrix
5. Systems of linear equations – Cramer’s rule
6. Methods of solving systems of equations, Gaussian elimination method
7. Eigenvalues and eigenvectors
8. Generalized inverse matrix (optional / to be confirmed)

Rectangular and Square Matrices

Definition of a Matrix

Definition:

A matrix is any array of numbers arranged in rows and columns. If a matrix has m rows and n columns, we say that it has dimension $m \times n$ *and we write:*

$$\mathbf{A} = [a_{ij}], \quad i = 1, 2, \ldots, m, \quad j = 1, 2, \ldots, n$$

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \ldots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \ldots & a_{2n} \\ . & . & . & \ldots & . \\ a_{m1} & a_{m2} & a_{m3} & \ldots & a_{mn} \end{bmatrix}$$

Rectangular and Square Matrices
Column and Row Vectors of a Matrix

Definition:

A *column vector* of a matrix **A** of dimension $m \times n$ is any submatrix of dimension $m \times 1$.
Every matrix can be divided into n column vectors.

$$\mathbf{A} = \left[\begin{array}{c|c|c|c|c} a_{11} & a_{12} & a_{13} & \ldots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \ldots & a_{2n} \\ . & . & . & \ldots & . \\ a_{m1} & a_{m2} & a_{m3} & \ldots & a_{mn} \end{array}\right]$$

Definition:

A *row vector* of a matrix **A** of dimension $m \times n$ is any submatrix of dimension $1 \times n$.
Every matrix can be divided into m row vectors.

$$\mathbf{A} = \left[\begin{array}{ccccc} a_{11} & a_{12} & a_{13} & \ldots & a_{1n} \\ \hline a_{21} & a_{22} & a_{23} & \ldots & a_{2n} \\ \hline . & . & . & \ldots & . \\ \hline a_{m1} & a_{m2} & a_{m3} & \ldots & a_{mn} \end{array}\right]$$

Rectangular and Square Matrices
Zero Matrix and Square Matrix

Definition:

A *zero matrix* of dimension $m \times n$ is a matrix consisting entirely of zeros.

That is, if $\boldsymbol{A} = \mathbf{0}_{m\times n}$, then $a_{ij} = 0$ for all i and j.

$$\mathbf{A} = \mathbf{0}_{mxn} = \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ . & . & \dots & . \\ 0 & 0 & \dots & 0 \end{bmatrix}$$

Definition:

If the number of rows equals the number of columns ($m=n$),then such an array of numbers is called a square matrix.

The elements $a_{11}, a_{22}, \ldots, a_{nn}$ form the *diagonal* of the matrix.

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ . & . & \dots & . \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

Rectangular and Square Matrices
Upper Triangular and Lower Triangular Matrices

Definition:

A square matrix such that a_{ij}=0 for $i>j$ is called *an upper triangular* matrix.

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2n} \\ . & . & \dots & . \\ 0 & 0 & \dots & a_{nn} \end{bmatrix}$$

Definition:

A square matrix such that a_{ij}=0 for $i<j$ is called *a lower triangular* matrix.

$$\begin{bmatrix} a_{11} & 0 & \dots & 0 \\ a_{21} & a_{22} & \dots & 0 \\ . & . & \dots & . \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

Rectangular and Square Matrices
Diagonal and Identity Matrices

Definition:

A square matrix such that $a_{ij}=0$ for $i\neq j$ is called a diagonal matrix.

$$\begin{bmatrix} a_{11} & 0 & \dots & 0 \\ 0 & a_{22} & \dots & 0 \\ . & . & \dots & . \\ 0 & 0 & \dots & a_{nn} \end{bmatrix}$$

Definition:

Every diagonal matrix in which $a_{ii}=1$ is called an identity matrix and is denoted by **I** or $\mathbf{I}_n$

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ . & . & \dots & . \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

Rectangular and Square Matrices

Matrix Equality

Two matrices **A** and **B** of the same dimension $m \times n$ are said to be equal if and only if all corresponding elements are equal, that is,

$$\mathbf{A} = \mathbf{B} \quad \Leftrightarrow \quad a_{ij} = b_{ij} \quad \text{for all } i, j.$$

Addition of Matrices

Let **A** = $[a_{ij}]$ and **B** = $[b_{ij}]$ be matrices of dimension $m \times n$. The sum of matrices **A** and **B** is the matrix **C**=$[c_{ij}]$, denoted **C**=**A**+**B**, whose elements are defined by:

$$c_{ij} = a_{ij} + b_{ij}, \quad i = 1, 2, \ldots, m, \; j = 1, 2, \ldots, n.$$

Rectangular and Square Matrices

Addition of Matrices

$$\mathbf{A}+\mathbf{B}=\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ . & . & \dots & . \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}+\begin{bmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ . & . & \dots & . \\ b_{m1} & b_{m2} & \dots & b_{mn} \end{bmatrix}=\begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \dots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \dots & a_{2n}+b_{2n} \\ . & . & \dots & . \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \dots & a_{mn}+b_{mn} \end{bmatrix}$$

$$\mathbf{A}+\mathbf{B}=\begin{bmatrix} 1 & 0 & -2 & 4 \\ 5 & 2 & 1 & 0 \\ -3 & 4 & 1 & 2 \end{bmatrix}+\begin{bmatrix} 2 & -1 & 3 & 1 \\ -2 & 1 & 0 & 2 \\ 0 & 2 & 5 & 3 \end{bmatrix}=$$

$$=\begin{bmatrix} 1+2 & 0+(-1) & -2+3 & 4+1 \\ 5+(-2) & 2+1 & 1+0 & 0+2 \\ -3+0 & 4+2 & 1+5 & 2+3 \end{bmatrix}=\begin{bmatrix} 3 & -1 & 1 & 5 \\ 3 & 3 & 1 & 2 \\ -3 & 6 & 6 & 5 \end{bmatrix}$$

Rectangular and Square Matrices

Subtraction of Matrices

Subtraction of Matrices

Let **A** = [a_{ij}] and **B** = [b_{ij}] be matrices of dimension $m \times n$. The *difference* of matrices **A** and **B** is the matrix **C**=[c_{ij}], denoted **C**=**A**−**B**, whose elements are defined by:

$$c_{ij} = a_{ij} - b_{ij}, \quad i = 1, 2, \ldots, m, \ j = 1, 2, \ldots, n.$$

Rectangular and Square Matrices

Subtraction of Matrices

$$\mathbf{A}-\mathbf{B}=\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ . & . & \dots & . \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}-\begin{bmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ . & . & \dots & . \\ b_{m1} & b_{m2} & \dots & b_{mn} \end{bmatrix}=\begin{bmatrix} a_{11}-b_{11} & a_{12}-b_{12} & \dots & a_{1n}-b_{1n} \\ a_{21}-b_{21} & a_{22}-b_{22} & \dots & a_{2n}-b_{2n} \\ . & . & \dots & . \\ a_{m1}-b_{m1} & a_{m2}-b_{m2} & \dots & a_{mn}-b_{mn} \end{bmatrix}$$

$$\mathbf{A}-\mathbf{B}=\begin{bmatrix} 1 & 0 & -2 & 4 \\ 5 & 2 & 1 & 0 \\ -3 & 4 & 1 & 2 \end{bmatrix}-\begin{bmatrix} 2 & -1 & 3 & 1 \\ -2 & 1 & 0 & 2 \\ 0 & 2 & 5 & 3 \end{bmatrix}=$$

$$=\begin{bmatrix} 1-2 & 0-(-1) & -2-3 & 4-1 \\ 5-(-2) & 2-1 & 1-0 & 0-2 \\ -3-0 & 4-2 & 1-5 & 2-3 \end{bmatrix}=\begin{bmatrix} -1 & 1 & -5 & 3 \\ 7 & 1 & 1 & -2 \\ -3 & 2 & -4 & -1 \end{bmatrix}$$

Rectangular and Square Matrices

Multiplication of a Matrix by a Scalar

Multiplication of a Matrix by a Scalar

Let **A** = $[a_{ij}]$ be a matrix of dimension $m \times n$, and let c be a real number (a scalar). The product of the matrix **A** by the scalar c is the matrix **B** = $[b_{ij}]$, (denoted **B** = c**A**), whose elements are defined by:

$$b_{ij} = c \cdot a_{ij}, \quad i = 1, 2, \ldots, m, \; j = 1, 2, \ldots, n.$$

$$c\mathbf{A} = 3\begin{bmatrix} 1 & 0 & -2 & 4 \\ 5 & 2 & 1 & 0 \\ -3 & 4 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 0 & -6 & 12 \\ 15 & 6 & 3 & 0 \\ -9 & 12 & 3 & 6 \end{bmatrix}$$

Properties of Matrix Addition and Scalar Multiplication

Let $\mathbf{A}, \mathbf{B}, \mathbf{C}$ be matrices of the same dimension $m \times n$,

and let $c, d \in \mathbb{R}$ be real numbers.

The following properties hold:

1. **Commutativity:**

   $A + B = B + A$

2. **Associativity:**

   $(A + B) + C = A + (B + C)$

3. **Additive Identity:**

   $A + 0 = A$

4. **Additive Inverse:**

   $A + (-A) = 0$

5. **Distributivity (Matrix Addition):**

   $c(A + B) = cA + cB$

6. **Distributivity (Scalar Addition):**

   $(c + d)A = cA + dA$

7. **Associativity (Scalars):**

   $(cd)A = c(dA)$

8. **Multiplicative Identity:**

   $1 \cdot A = A$

Rectangular and Square Matrices
Matrix Multiplication

Matrix Multiplication A · B = AB = C:

The product A·B is defined only if the number of columns in A equals the number of rows in B.

Let **A**=[a_{ij}] be a matrix of dimension $m \times n$ and

B=[b_{jk}] a matrix of dimension $n \times p$.

The *product* of matrices **A** and **B** is a matrix **C**=[c_{ik}] of dimension m x p, defined as:

$$\mathbf{C} = \mathbf{AB}, \qquad c_{ik} = a_{i1}b_{1k} + a_{i2}b_{2k} + \ldots + a_{in}b_{nk} = \sum_{j=1}^{n} a_{ij}b_{jk}$$

Remarks:

- Matrix multiplication **is not commutative**, i.e. in general
  $\mathbf{AB} \neq \mathbf{BA}$.
- However, for some special matrices (e.g. diagonal or scalar multiples of the identity), commutativity may hold.

Rectangular and Square Matrices

Matrix Multiplication — Example

$$\mathbf{AB} = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1\cdot 3 + 2\cdot 1 & 1\cdot 1 + 2\cdot 2 \\ 0\cdot 3 + (-1)\cdot 1 & 0\cdot 1 + (-1)\cdot 2 \end{bmatrix} = \begin{bmatrix} 5 & 5 \\ -1 & -2 \end{bmatrix}$$

$$\mathbf{AB} = \begin{bmatrix} 1 & 0 & -1 \\ 5 & 2 & 3 \end{bmatrix} \cdot \begin{bmatrix} 2 & -1 & 3 & 1 \\ -2 & 1 & 0 & 2 \\ 0 & 2 & 5 & 3 \end{bmatrix} =$$

$$= \begin{bmatrix} 1\cdot 2 + 0\cdot(-2) + (-1)\cdot 0 & 1\cdot(-1) + 0\cdot 1 + (-1)\cdot 2 & 1\cdot 3 + 0\cdot 0 + (-1)\cdot 5 & 1\cdot 1 + 0\cdot 2 + (-1)\cdot 3 \\ 5\cdot 2 + 2\cdot(-2) + 3\cdot 0 & 5\cdot(-1) + 2\cdot 1 + 3\cdot 2 & 5\cdot 3 + 2\cdot 0 + 3\cdot 5 & 5\cdot 1 + 2\cdot 2 + 3\cdot 3 \end{bmatrix} =$$

$$= \begin{bmatrix} 2 & -3 & -2 & -2 \\ 6 & 3 & 30 & 18 \end{bmatrix}$$

Rectangular and Square Matrices
Matrix Transposition

Matrix Transposition

The *transpose* of a matrix **A**, denoted – $\mathbf{A}^T$ is the matrix obtained by interchanging the rows and columns of **A**.

$$(\mathbf{A}^T)_{ij} = a_{ji}, \quad \text{for all } i, j.$$

Example:

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & -2 & 4 \\ 5 & 2 & 1 & 0 \\ -3 & 4 & 1 & 2 \end{bmatrix} \qquad \mathbf{A}^T = \begin{bmatrix} 1 & 5 & -3 \\ 0 & 2 & 4 \\ -2 & 1 & 1 \\ 4 & 0 & 2 \end{bmatrix}$$

Rectangular and Square Matrices
Properties of Matrix Operations

$$\mathbf{A}(\mathbf{B}+\mathbf{C}) = \mathbf{AB}+\mathbf{AC} \qquad (\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$$

$$(\mathbf{A}+\mathbf{B})\mathbf{C} = \mathbf{AC}+\mathbf{BC} \qquad \mathbf{AI}_{\mathrm{n}} = \mathbf{I}_{\mathrm{m}}\mathbf{A} = \mathbf{A}$$

$$\mathbf{A}(\mathrm{c}\mathbf{B}) = (\mathrm{c}\mathbf{A})\mathbf{B} = \mathrm{c}\mathbf{AB} \qquad \mathbf{A0}_{\mathrm{nxp}} = \mathbf{0}_{\mathrm{mxp}}$$

$$(\mathbf{A}^T)^T = \mathbf{A},$$
$$(\mathbf{A}+\mathbf{B})^T = \mathbf{A}^T+\mathbf{B}^T,$$
$$(c\mathbf{A})^T = c\mathbf{A}^T,$$
$$(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T.$$

Square Matrices
Trace of a Matrix

Trace of a Matrix

The trace of a square matrix **A**, denoted tr(**A**), is the sum of the elements on its main diagonal:

$$\mathrm{tr}(\mathbf{A}) = \sum_{i=1}^{n} a_{ii}.$$

$$\mathrm{tr}(\mathbf{A}) = \mathrm{tr}\left(\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ . & . & \cdots & . \\ a_{m1} & a_{m2} & \cdots & a_{mm} \end{bmatrix}\right) = \sum_{i=1}^{m} a_{ii} = a_{11} + a_{22} + \ldots + a_{mm}$$

$$\mathrm{tr}(\mathbf{A}) = \mathrm{tr}\left(\begin{bmatrix} 1 & -2 & 1 \\ 3 & 2 & 3 \\ -1 & 4 & 3 \end{bmatrix}\right) = 1 + 2 + 3 = 6$$

Square Matrices

Trace of a Matrix

$$tr(c\mathbf{A}) = c \cdot tr(\mathbf{A}) \qquad tr(\mathbf{A}+\mathbf{B}) = tr(\mathbf{A}) + tr(\mathbf{B}) \qquad tr(\mathbf{A}) = tr(\mathbf{A}^T)$$

$$\mathrm{tr}(4\mathbf{A}) = \mathrm{tr}(4\begin{bmatrix} 1 & -2 & 1 \\ 3 & 2 & 3 \\ -1 & 4 & 3 \end{bmatrix}) = \mathrm{tr}(\begin{bmatrix} 4 & -8 & 4 \\ 12 & 8 & 12 \\ -4 & 16 & 12 \end{bmatrix}) = 4 \cdot 6 = 24$$

$$\mathrm{tr}(\mathbf{A}) = \mathrm{tr}(\begin{bmatrix} 1 & -2 & 1 \\ 3 & 2 & 3 \\ -1 & 4 & 3 \end{bmatrix}) = 6 = \mathrm{tr}(\mathbf{A}^{\mathrm{T}}) = \mathrm{tr}(\begin{bmatrix} 1 & 3 & -1 \\ -2 & 2 & 4 \\ 1 & 3 & 3 \end{bmatrix})$$

$$\mathrm{tr}(\mathbf{A}+\mathbf{B}) = \mathrm{tr}(\mathbf{A}) + \mathrm{tr}(\mathbf{B}) = \mathrm{tr}(\begin{bmatrix} 1 & -2 & 1 \\ 3 & 2 & 3 \\ -1 & 4 & 3 \end{bmatrix}) + \mathrm{tr}(\begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix}) = 6 + 3 = 9$$

Linearly Independent Vectors

Definition:

A family of vectors $\mathbf{a}_1$, $\mathbf{a}_2$, …, $\mathbf{a}_k$ is said to be *linearly independent* if the only solution of the equation

$$c_1\mathbf{a_1} + c_2\mathbf{a_2} + \cdots + c_k\mathbf{a_k} = \mathbf{0}$$

is

$$c_1 = c_2 = \cdots = c_k = 0.$$

Note:

If the above equation also has a non-zero solution, then the vectors $\mathbf{a}_1$, $\mathbf{a}_2$, …, $\mathbf{a}_k$ are *linearly dependent.*

Rectangular and Square Matrices

Rank of a Matrix and Non-Singular Matrices

Definition:

The rank of a matrix A is the maximum number of linearly independent vectors(rows or columns) in the matrix. It is denoted by rank **A** or r_a = rz **A**

Definition:

The rank of a matrix is equal to the highest order of any non-zero determinant of its square submatrices (i.e. the highest order of its non-singular minor).
(see L. 3 and 4).

In practice, the rank of a matrix **A** is found by transforming it into its *echelon form* using *elementary operations*.

Rectangular and Square Matrices
Echelon (Canonical) Form of a Matrix

The *echelon form* (or row-echelon form) of a matrix **A** is obtained by performing a sequence of elementary transformations on the rows (and possibly columns) of **A**.

In echelon form:

- all zero rows, if any, appear at the bottom of the matrix,
- the leading non-zero entry in each non-zero row lies to the right of the leading entry in the row above.

$$\mathbf{A} = \begin{bmatrix} \mathbf{I}_k & \mathbf{R} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \Rightarrow \text{rz}\mathbf{A} = \text{k}$$

$$\mathbf{A} = \begin{bmatrix} 5 & 2 & 1 & -2 \\ -1 & 3 & 0 & -3 \\ 0 & 2 & -1 & -2 \\ 1 & -3 & 0 & 3 \\ 10 & 4 & 2 & -4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \mathbf{I}_3 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \Rightarrow rz\mathbf{A} = 3$$

Elementary Transformations Used to Determine the Rank

1. Add to one row another row multiplied by a non-zero scalar:

   $W_i \to W_i + cW_j$

2. Add to one column another column multiplied by a non-zero scalar:

   $K_i \to K_i + cK_j$

3. Multiply a row by a non-zero scalar:

   $W_i \to cW_i$

4. Multiply a column by a non-zero scalar:

   $K_i \to cK_i$

5. Interchange two rows:

   $W_i \leftrightarrow W_j$

6. Interchange two columns:

   $K_i \leftrightarrow K_j$

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Rectangular and Square Matrices
Rank of a Matrix — Example

Example:

$$rz\mathbf{A} = rz\begin{bmatrix} 5 & 2 & 1 \\ -1 & 3 & 0 \\ 0 & 2 & -1 \end{bmatrix} = ?$$

$$\Rightarrow rz\mathbf{A} = 3$$

Rectangular and Square Matrices

Linearly Independent Vectors

In R^3

$$rz\mathbf{A} = rz\begin{bmatrix} 5 & 2 & 1 \\ -1 & 3 & 0 \\ 0 & 2 & -1 \end{bmatrix} = 3$$

Since rank **A** = 3, both the column vectors and the row vectors of matrix **A** are linearly independent.

$$\operatorname{rank}(\mathbf{A}) = 3 \quad \Rightarrow \quad c_1 = c_2 = c_3 = 0$$

$$\mathbf{a_1} = \begin{bmatrix} 5 \\ -1 \\ 0 \end{bmatrix}, \mathbf{a}_2 = \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}, \mathbf{a}_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$c_1\mathbf{a_1} + c_2\mathbf{a}_2 + c_3\mathbf{a}_3 = \mathbf{0}$$

$$c_1\begin{bmatrix} 5 \\ -1 \\ 0 \end{bmatrix} + c_2\begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix} + c_3\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \mathbf{0}$$

Rectangular and Square Matrices

Echelon (Canonical) Form of a Matrix

Example:

$$rz\mathbf{A} := rz\begin{bmatrix} 1 & 2 & 3 & -3 \\ -1 & -3 & 5 & 3 \\ 3 & 6 & 9 & -9 \\ 1 & 1 & 11 & -3 \end{bmatrix} \overset{W_2 \to W_2 + W_1}{=} rz\begin{bmatrix} 1 & 2 & 3 & -3 \\ 0 & -1 & 8 & 0 \\ 3 & 6 & 9 & -9 \\ 1 & 1 & 11 & -3 \end{bmatrix} \overset{W_3 \to W_3 - 3W_1}{=}$$

$$= rz\begin{bmatrix} 1 & 2 & 3 & -3 \\ 0 & -1 & 8 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 11 & -3 \end{bmatrix} \overset{W_4 \to W_4 - W_1}{=} rz\begin{bmatrix} 1 & 2 & 3 & -3 \\ 0 & -1 & 8 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 8 & 0 \end{bmatrix} \overset{W_4 \to W_4 - W_2}{=}$$

$$= rz\begin{bmatrix} 1 & 2 & 3 & -3 \\ 0 & -1 & 8 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \overset{\substack{K_2 \to K_2 - 2K_1 \\ K_2 \to (-1)K_2}}{=} rz\left[\begin{array}{cc|cc} 1 & 0 & 3 & -3 \\ 0 & 1 & 8 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] = rz\begin{bmatrix} \mathbf{I} & \mathbf{R} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \Rightarrow rz\mathbf{A} = 2$$

Square Matrices

Symmetric Matrix :

A *symmetric* matrix is a square matrix A such that $\mathbf{A}=\mathbf{A}^T$. That is, the matrix remains unchanged when transposed.

$$\mathbf{A} = \begin{bmatrix} a & d & e \\ d & b & f \\ e & f & c \end{bmatrix} = \mathbf{A}^T \qquad \mathbf{A} + \mathbf{A}^T = ?$$

Skew-Symmetric Matrix :

A *skew-symmetric* matrix (also called antisymmetric)is a square matrix A satisfying the relation $\mathbf{A}=-\mathbf{A}^T$.

$$\mathbf{A} = \begin{bmatrix} 0 & d & e \\ -d & 0 & f \\ -e & -f & 0 \end{bmatrix} = -\mathbf{A}^T \qquad \mathbf{A} - \mathbf{A}^T = ?$$

Remark:

For every skew-symmetric matrix, all elements on the main diagonal are equal to zero.

Square Matrices

Hilbert Matrix:

A Hilbert matrix is a symmetric square matrix **A**=[1/(i+j)], where
i, j = 1,2,…,*m*

$$\mathbf{A} = \begin{bmatrix} \frac{1}{1+1} & \frac{1}{1+2} & \frac{1}{1+3} \\ \frac{1}{2+1} & \frac{1}{2+2} & \frac{1}{2+3} \\ \frac{1}{3+1} & \frac{1}{3+2} & \frac{1}{3+3} \end{bmatrix} = \mathbf{A}^{T}$$

Commutative Matrices

Two square matrices **A** and **B** are said to be commutative if their product does not depend on the order of multiplication, i.e.

$$\mathbf{AB} = \mathbf{BA}.$$