Mathematics I

Land Resources Management

Prof. UPP Agnieszka Łacka, PhD, DSc
Department of Mathematical and Statistical Methods
Room 710
Poznań University of Life Sciences

Course Program – Linear Algebra

1. Vector spaces, vectors, and unit vectors (versors)
2. Types of matrices, operations on matrices, matrix rank
3. **Determinants and their properties**
4. **Inverse matrix**
5. Systems of linear equations – Cramer’s rule
6. Methods of solving systems of equations, Gaussian elimination method
7. Eigenvalues and eigenvectors
8. Generalized inverse matrix (optional / to be confirmed)

ATTENTION!

The concept of a determinant applies only to square matrices!!!

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 11 & 2 & 1 \\ 5 & 3 & 12 & 3 & 0 \\ 4 & 2 & 14 & 12 & 1 \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 4 & 2 & 1 \end{bmatrix}$$

Determinants and their properties
Definition of the determinant

Definition :

The *determinant* of a square matrix is a function that assigns a real number **det A** to every real square matrix **A**=[a_{ij}]. This function is defined recursively as follows:

1. If **A** is of order $n = 1$, then det **A** = a_{11}

2. If **A** is of order n, then

$$\det \mathbf{A} = a_{i1} D_{i1} + a_{i2} D_{i2} + \ldots + a_{in} D_{in}$$

or

$$\det \mathbf{A} = a_{1j} D_{1j} + a_{2j} D_{2j} + \ldots + a_{nj} D_{nj}$$

where $1 \le i, j \le n$, and the symbol D_{ij} denotes the *algebraic cofactor* of the element a_{ij}.

Determinants and their properties
Definition of the algebraic cofactor

Definition :

The *algebraic cofactor* of the element a_{ij} of matrix **A** is the number

$$D_{ij}=(-1)^{i+j}\det \mathbf{A}_{ij},$$

where $\mathbf{A}_{ij}$ is the matrix of order $n-1$ obtained from **A** by deleting the i-th row and the j-th column.

The determinant $\det \mathbf{A}_{ij}$ is called the i j-th *minor*.

$$\mathbf{A}=\begin{bmatrix} 5 & \textcircled{2} & 1 \\ -1 & 3 & 0 \\ 0 & 2 & -1 \end{bmatrix}$$

$$D_{12}=(-1)^{1+2}\det \mathbf{A}_{12}=(-1)^{1+2}\det\begin{bmatrix} 5 & \textcircled{2} & 1 \\ -1 & 3 & 0 \\ 0 & 2 & -1 \end{bmatrix}=(-1)^{3}\det\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

Determinants and their properties

Methods of calculating the determinant

Methods of calculating the determinant:

› For a first-order matrix – directly from the definition.

› Determinant of a second-order matrix (n = 2).

› Laplace expansion:

– with respect to the i-th row $\det \mathbf{A} = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det \mathbf{A}_{ij}$

– with respect to the j-th column. $\det \mathbf{A} = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det \mathbf{A}_{ij}$

› **Sarrus' rule** – used for determining the determinant of a third-order matrix.

› Methods using the properties of the determinant function det **A.**

Methods of calculating the determinant:

- n=1 If **A** is of order n = 1, then det**A** = a_{11}

$$\mathbf{A} = [a_{11}] = [5] \Rightarrow \det \mathbf{A} = |5| = 5$$

- n=2 For n = 2, the determinant is calculated according to the formula:

$$\det \mathbf{A} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ -1 & 3 \end{bmatrix} \quad \Rightarrow \quad \det \mathbf{A} = \begin{vmatrix} 5 & 2 \\ -1 & 3 \end{vmatrix} = 5 \cdot 3 - 2 \cdot (-1) = 15 + 2 = 17$$

Methods of calculating the determinant:

- n ≥ 2 Laplace expansion
  - with respect to the i-th row $$\det \mathbf{A} = \sum_{j=1}^{n} a_{ij} D_{ij} = a_{i1} D_{i1} + a_{i2} D_{i2} + \ldots + a_{in} D_{in}$$
  - with respect to the j-th column $$\det \mathbf{A} = \sum_{i=1}^{n} a_{ij} D_{ij} = a_{1j} D_{1j} + a_{2j} D_{2j} + \ldots + a_{nj} D_{nj}$$

Example. Laplace expansion with respect to the first row (i = 1) of matrix **A**

$$\det \mathbf{A} = \begin{vmatrix} 5 & 2 & 1 \\ -1 & 3 & 0 \\ 0 & 2 & -1 \end{vmatrix} = 5D_{11} + 2D_{12} + 1D_{13} =$$

$$= 5(-1)^{1+1} \begin{vmatrix} 3 & 0 \\ 2 & -1 \end{vmatrix} + 2(-1)^{1+2} \begin{vmatrix} -1 & 0 \\ 0 & -1 \end{vmatrix} + 1(-1)^{1+3} \begin{vmatrix} -1 & 3 \\ 0 & 2 \end{vmatrix} =$$

$$= 5(-3-0) - 2(1-0) + (-2-0) = -15 - 2 - 2 = -19$$

Example. Laplace expansion with respect to the second column (j = 2) of matrix **A**

$$\det \mathbf{A} = \begin{vmatrix} 5 & 2 & 1 \\ -1 & 3 & 0 \\ 0 & 2 & -1 \end{vmatrix} = 2D_{12} + 3D_{22} + 2D_{32} =$$

$$= -19$$

Determinants and their properties
Sarrus' rule

Sarrus' rule is used to calculate the determinant of a third-order matrix **A** (n=3) :

$$\det \mathbf{A} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{matrix} =$$

$$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} +$$

$$-(a_{13}a_{22}a_{31} + a_{11}a_{23}a_{32} + a_{12}a_{21}a_{33})$$

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Determinants and their properties
Example – Sarrus' rule

Sarrus' rule is used to calculate the determinant of a third-order matrix **A** (n=3) :

$$\det \mathbf{A} = \begin{vmatrix} 5 & 2 & 1 \\ -1 & 3 & 0 \\ 0 & 2 & -1 \end{vmatrix}\begin{matrix} 5 & 2 \\ -1 & 3 \\ 0 & 2 \end{matrix} =$$

$$= 5\cdot 3\cdot(-1) + 2\cdot 0\cdot 0 + 1\cdot(-1)\cdot 2 +$$

$$-(1\cdot 3\cdot 0 + 5\cdot 0\cdot 2 + 2\cdot(-1)\cdot(-1)) =$$

$$= -15 + 0 - 2 - (0 + 0 + 2) = -19$$

(The detailed numerical computation follows the Sarrus diagram — diagonals drawn through elements of matrix **A** to compute positive and negative products.)

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 5 & 2 & 1 \\ -1 & 3 & 0 \\ 0 & 2 & -1 \end{bmatrix} \Rightarrow \det \mathbf{A} = \begin{vmatrix} 5 & 2 & 1 \\ -1 & 3 & 0 \\ 0 & 2 & -1 \end{vmatrix} = -19$$

Determinants and their properties
Nine properties of the determinant

1. The determinant of a square matrix that contains a column (or a row) consisting entirely of zeros is equal to 0;

2. The determinant of a square matrix that has two identical columns (or rows) is equal to 0;

$$\det \mathbf{A} = \begin{vmatrix} 5 & 2 & 0 \\ -1 & 3 & 0 \\ 0 & 2 & 0 \end{vmatrix} \begin{matrix} 5 & 2 \\ -1 & 3 \\ 0 & 2 \end{matrix} =$$

$$= 5\cdot 3\cdot (0) + 2\cdot (0)\cdot 0 + (0)\cdot (-1)\cdot 2 +$$

$$-((0)\cdot 3\cdot 0 + 5\cdot (0)\cdot 2 + 2\cdot (-1)\cdot (0)) = 0$$

$$\det \mathbf{A} = \begin{vmatrix} 5 & 2 & 2 \\ -1 & 3 & 3 \\ 0 & 2 & 2 \end{vmatrix} \begin{matrix} 5 & 2 \\ -1 & 3 \\ 0 & 2 \end{matrix} =$$

$$= \underline{5\cdot 3\cdot 2} + \underline{2\cdot 3\cdot 0} + 2\cdot (-1)\cdot 2 +$$

$$-(\underline{2\cdot 3\cdot 0} + \underline{5\cdot 3\cdot 2} + 2\cdot (-1)\cdot 2) = 0$$

Determinants and their properties
Nine properties of the determinant

3. The determinant of a square matrix changes its sign if two columns (or two rows) are interchanged;

4. If all elements of a certain column (or row) of a square matrix contain a common factor, this factor may be factored out in front of the determinant of that matrix;

$$\det \mathbf{A} = \begin{vmatrix} 5 & 2 & 1 \\ -1 & 3 & 0 \\ 0 & 2 & -1 \end{vmatrix} = -15 - 2 - 2 = -19$$

$$\det \mathbf{A} = \begin{vmatrix} 5 & 1 & 2 \\ -1 & 0 & 3 \\ 0 & -1 & 2 \end{vmatrix} \begin{matrix} 5 & 1 \\ -1 & 0 \\ 0 & -1 \end{matrix} =$$

$$= 0 + 0 + 2 - (0 - 15 - 2) = 19$$

$$\det \mathbf{A} = \begin{vmatrix} 5 & 2 & 1 \\ -1 & 2 & 0 \\ 0 & 2 & -1 \end{vmatrix} = -10 - 2 - 2 = -14$$

$$\det \mathbf{A} = 2\begin{vmatrix} 5 & 1 & 1 \\ -1 & 1 & 0 \\ 0 & 1 & -1 \end{vmatrix} = 2(-5 - 1 - 1) = -14$$

Determinants and their properties

Nine properties of the determinant

5. The determinant of a square matrix in which the elements of one column (or row) are sums of two components is equal to the sum of the determinants of matrices obtained by replacing that column (or row) with the respective components.

$$\det \mathbf{A} = \begin{vmatrix} 5 & 2 & 1 \\ -1 & 3 & 0 \\ 0 & 2 & -1 \end{vmatrix} =$$

$$= \begin{vmatrix} 5 & 1 & 1 \\ -1 & 1 & 0 \\ 0 & 1 & -1 \end{vmatrix} + \begin{vmatrix} 5 & 1 & 1 \\ -1 & 2 & 0 \\ 0 & 1 & -1 \end{vmatrix} =$$

$$= (-5-1-1) + (-10-1-1) = -19$$

Determinants and their properties
Nine properties of the determinant

6. The determinant of a matrix does not change if we add to the elements of any column (or row) the corresponding elements of another column (or row) multiplied by any real number;

$$\det \mathbf{A} = \begin{vmatrix} 5 & 2 & 1 \\ -1 & 3 & 0 \\ 0 & 2 & -1 \end{vmatrix} = -19 \qquad W_1 \rightarrow W_1 + W_3$$

$$\det \mathbf{A} = \begin{vmatrix} 5+0 & 2+2 & 1-1 \\ -1 & 3 & 0 \\ 0 & 2 & -1 \end{vmatrix} = \begin{vmatrix} 5 & 4 & 0 \\ -1 & 3 & 0 \\ 0 & 2 & -1 \end{vmatrix} =$$

$$= (-1)^{3+3}(-1)(15+4) = -19$$

Example notation:
$\mathbf{W_3 \rightarrow W_3 + cW_1}$
means: add to the third row ($\mathbf{W_3}$) the first row ($\mathbf{W_1}$) multiplied by a constant **c.**

Determinants and their properties
Nine properties of the determinant

7. The determinants of a matrix and its transpose are equal: det **A** = det $\mathbf{A}^T$

8. The determinant of a triangular matrix is equal to the product of its diagonal elements.

9. The determinant of a product of matrices is equal to the product of their determinants: det **AB** = det **A** det **B**

$$\det \mathbf{A} = \begin{vmatrix} 5 & 2 & 1 \\ 0 & 3 & -1 \\ 0 & 0 & 1 \end{vmatrix} = 15$$

$$\det \mathbf{B} = \begin{vmatrix} 5 & 0{,}03 & -710 \\ 0 & 3 & 14080 \\ 0 & 0 & 1 \end{vmatrix} = 15$$

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Determinants and their properties

Transforming a matrix into lower or upper triangular form

Elementary transformation operations are the same as those used for determining the rank of a matrix A:

Adding to a row another row multiplied by a nonzero number.

$W_i \rightarrow W_i + cW_j$ means adding to the i-th row the j-th row multiplied by a constant $c \neq 0$.

Adding to a column another column multiplied by a nonzero number.

$K_i \rightarrow K_i + cK_j$ means adding to the i-th column the j-th column multiplied by a constant $c \neq 0$.

Multiplying a column by a nonzero scalar.

$K_i \rightarrow cK_i$ means multiplying the i-th column by a scalar c.

Effect on the determinant: If one column is multiplied by c, then the determinant of the matrix is multiplied by c. In symbols: if matrix B is obtained from A by replacing column i by c times that column, then $\det\mathbf{B} = c\det\mathbf{A}$.

Multiplying a row by a nonzero scalar.

$W_i \rightarrow cW_i$ means multiplying the i-th row by a scalar c.

Effect on the determinant: If one row is multiplied by c, then the determinant is multiplied by c.

Determinants and their properties

Example

$$\det \mathbf{A} = \begin{vmatrix} 5 & 2 & 1 \\ -1 & 3 & 0 \\ 0 & 2 & -1 \end{vmatrix} = -19$$

$$\det \mathbf{A} = \begin{vmatrix} 5 & 2 & 1 \\ -1 & 3 & 0 \\ 0 & 2 & -1 \end{vmatrix} \overset{W_1 \to W_1 + W_3}{=} \begin{vmatrix} 5 & 4 & 0 \\ -1 & 3 & 0 \\ 0 & 2 & -1 \end{vmatrix} \overset{K_2 \to K_2 - \frac{4}{5}K_1}{=}$$

$$= \begin{vmatrix} 5 & 0 & 0 \\ -1 & \frac{4}{5}+3 & 0 \\ 0 & 2 & -1 \end{vmatrix} = 5 \cdot \frac{19}{5} \cdot (-1) = -19$$

Lower triangular matrix

Determinant and rank of a matrix
Definition of a non-singular matrix

Definition:

A *non-singular* matrix is a square matrix **A** for which det $\mathbf{A} \neq 0$.

Definition:

A *non-singular* matrix is an m x m square matrix A that is of full rank, that is, rz**A** = m.

Determinant and rank of a matrix
Definition of a minor

- **Definition** :

A *minor* is the determinant of a square submatrix.

$$\mathbf{A} = \begin{bmatrix} 1 & 5 & 2 & 1 & 1 \\ 2 & -1 & 3 & 0 & 1 \\ -3 & 0 & 2 & -1 & 4 \\ 0 & 3 & 0 & 1 & 3 \end{bmatrix}$$

One of many minors

$$\begin{vmatrix} 5 & 2 & 1 \\ -1 & 3 & 0 \\ 0 & 2 & -1 \end{vmatrix} = -19$$

- **Definition** :

The *rank* of a matrix A is the highest degree of a nonzero minor of that matrix.

Inverse matrix
Definition and properties

Definition :

The *inverse matrix* of a square matrix **A** is a matrix denoted by $\mathbf{A}^{-1}$, such that

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$$

where **I** is the identity matrix of the same order as **A**

Note :

Only **nonsingular matrices** (those for which det $\mathbf{A} \neq 0$) possess an inverse matrix.

Property :

The determinant of the inverse matrix is equal to the reciprocal of the determinant of the original matrix:

$$\det(\mathbf{A}^{-1}) = \frac{1}{\det(\mathbf{A})}$$

Inverse matrix

Methods of determining the inverse matrix

Methods of determining the inverse matrix:

- **From the matrix equation**. If $\mathbf{A}=[a_{ij}]$ is an $n \times n$ matrix, then $\mathbf{A}^{-1}=[x_{ij}]$ and the elements x_{ij} can be found from the equation $\mathbf{AA}^{-1} = \mathbf{I}_n$

- **From the formula:** $$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}}\left[D_{ij}\right]^T$$

  where $[D_{ij}]$ is the matrix of algebraic cofactors of **A**.

- **Using the algorithm of matrix inversion (Gauss–Jordan elimination method).**

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}}[D_{ij}]^T$$

Inverse matrix
Example 1

The algebraic cofactor of an element a_{ij} of matrix **A** is the numer $D_{ij}=(-1)^{i+j}\det \mathbf{A}_{ij}$, where $\mathbf{A}_{ij}$ is the submatrix obtained from **A** by deleting the i-th row and the j-th column.

$$\mathbf{A} = \begin{bmatrix} 5 & 2 & 1 \\ -1 & 3 & 0 \\ 0 & 2 & -1 \end{bmatrix}$$

$$D_{11} = (-1)^{1+1}\begin{vmatrix} 3 & 0 \\ 2 & -1 \end{vmatrix} = -3$$

$$D_{12} = (-1)^{1+2}\begin{vmatrix} -1 & 0 \\ 0 & -1 \end{vmatrix} = -1$$

$$D_{13} = (-1)^{1+3}\begin{vmatrix} -1 & 3 \\ 0 & 2 \end{vmatrix} = -2$$

itd

$$\det \mathbf{A} = -19$$

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}}[D_{ij}]^T = \frac{1}{(-19)}\begin{bmatrix} -3 & -1 & -2 \\ 4 & -5 & -10 \\ -3 & -1 & 17 \end{bmatrix}^T =$$

$$= \frac{-1}{19}\begin{bmatrix} -3 & 4 & -3 \\ -1 & -5 & -1 \\ -2 & -10 & 17 \end{bmatrix}$$

Verification:

$$\mathbf{AA}^{-1} = \frac{-1}{19}\begin{bmatrix} 5 & 2 & 1 \\ -1 & 3 & 0 \\ 0 & 2 & -1 \end{bmatrix}\begin{bmatrix} -3 & 4 & -3 \\ -1 & -5 & -1 \\ -2 & -10 & 17 \end{bmatrix} = \mathbf{I}$$

Inverse matrix
Example 2

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \left[D_{ij} \right]^T$$

Consider a square matrix **A** of order 2.

Check whether the matrix is nonsingular.

If it is nonsingular (det A ≠ 0), compute its inverse matrix $\mathbf{A}^{-1}$

$$\mathbf{A} = \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix}$$

$$D_{11} = (-1)^{1+1}|2| = 2$$

$$D_{12} = (-1)^{1+2}|-1| = (-1)(-1) = 1$$

$$\det \mathbf{A} = \begin{vmatrix} 2 & 3 \\ -1 & 2 \end{vmatrix} = 4 - (-3) = 7$$

$$D_{21} = (-1)^{2+1}|3| = (-1) \cdot 3 = -3$$

$$D_{22} = (-1)^{2+2}|2| = 2$$

Inverse matrix
Example 2

$$\mathbf{A} = \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix}$$

$$\det \mathbf{A} = \begin{vmatrix} 2 & 3 \\ -1 & 2 \end{vmatrix} = 4 - (-3) = 7$$

$$D_{11} = (-1)^{1+1}|2| = 2$$

$$D_{12} = (-1)^{1+2}|-1| = (-1)(-1) = 1$$

$$D_{21} = (-1)^{2+1}|3| = (-1)\cdot 3 = -3$$

$$D_{22} = (-1)^{2+2}|2| = 2$$

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}}\left[D_{ij}\right]^T = \frac{1}{\det \mathbf{A}}\begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix}^T$$

$$= \frac{1}{7}\begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix}^T = \frac{1}{7}\begin{bmatrix} 2 & -3 \\ 1 & 2 \end{bmatrix}$$

Verification:

$$\mathbf{A}^{-1}\mathbf{A} = \frac{1}{7}\begin{bmatrix} 2 & -3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix} =$$

$$= \frac{1}{7}\begin{bmatrix} 4+3 & 6-6 \\ 2-2 & 3+4 \end{bmatrix} = \mathbf{I}$$

Inverse matrix
Gauss–Jordan elimination method

- **Theorem :**

  If a square matrix **A** is **nonsingular** (det **A** $\neq$ 0), then there exists a sequence of elementary row transformations that reduces **A** to the identity matrix **I**.

- **Note :**

  If, during the sequence of elementary transformations, a row or column of zeros appears, it means that the inverse matrix $\mathbf{A}^{-1}$ does **not** exist (**A** is **singular**).

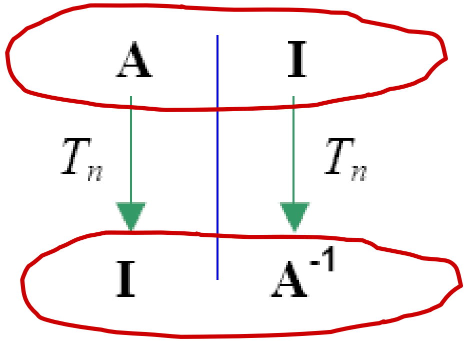
Inverse matrix
Gauss–Jordan elimination method

Matrix inversion algorithm (Gauss-Jordan elimination method)

Theorem:

If a sequence of elementary transformations reduces a nonsingular square matrix **A** of order *n* to the identity matrix $\mathbf{I}_n$, then applying the **same sequence** of elementary transformations to the identity matrix $\mathbf{I}_n$ produces the inverse matrix $\mathbf{A}^{-1}$.

(Transformations are performed **only on rows.**)



Inverse matrix
Gauss–Jordan elimination method

<u>Elementary transformations for calculating the inverse matrix</u>

1)Adding to a row another row multiplied by a nonzero scalar.

- $W_i \to W_i + cW_j$ means adding to the i-th row the j-th row multiplied by $c \neq 0$.

2)Multiplying a row by a nonzero scalar.

- $W_i \to cW_i$ means multiplying the i-th row by a scalar $c \neq 0$.

3)Interchanging two rows.

- $W_i \leftrightarrow W_j$ means swapping the i-th and j-th rows.

These transformations are applied successively to convert

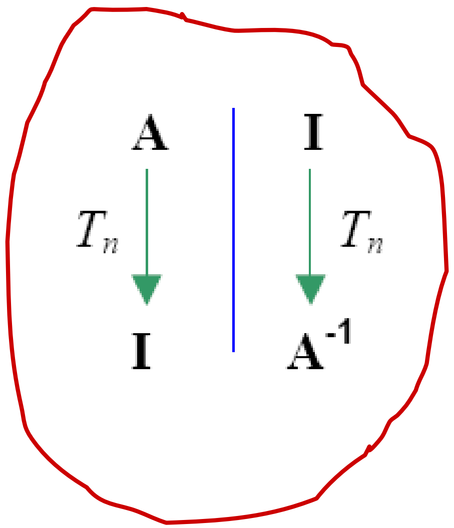
$$[\mathbf{A} \mid \mathbf{I}] \longrightarrow [\mathbf{I} \mid \mathbf{A}^{-1}]$$

Inverse matrix

Example 3

Transformations are applied only to the rows of the augmented matrix.

[A | I] **[I | A^{-1}]**



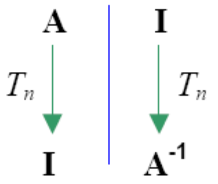
$$\left[\begin{array}{ccc|ccc} 5 & 2 & 1 & 1 & 0 & 0 \\ -1 & 3 & 0 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \to \cdots \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{3}{19} & -\frac{4}{19} & \frac{3}{19} \\ 0 & 1 & 0 & \frac{1}{19} & \frac{5}{19} & \frac{1}{19} \\ 0 & 0 & 1 & \frac{2}{19} & \frac{10}{19} & -\frac{17}{19} \end{array}\right]$$

$$\mathbf{A} = \begin{bmatrix} 5 & 2 & 1 \\ -1 & 3 & 0 \\ 0 & 2 & -1 \end{bmatrix} \to \mathbf{A}^{-1} = \begin{bmatrix} \frac{3}{19} & \frac{-4}{19} & \frac{3}{19} \\ \frac{1}{19} & \frac{5}{19} & \frac{1}{19} \\ \frac{2}{19} & \frac{10}{19} & \frac{-17}{19} \end{bmatrix} = \frac{-1}{19} \begin{bmatrix} -3 & 4 & -3 \\ -1 & -5 & -1 \\ -2 & -10 & 17 \end{bmatrix}$$

Verification :

$$\mathbf{A}^{-1}\mathbf{A} = \frac{-1}{19} \begin{bmatrix} -3 & 4 & -3 \\ -1 & -5 & -1 \\ -2 & -10 & 17 \end{bmatrix} \begin{bmatrix} 5 & 2 & 1 \\ -1 & 3 & 0 \\ 0 & 2 & -1 \end{bmatrix} = \mathbf{I}_3$$

Inverse matrix
Example 3



Matrix inversion algorithm (Gauss-Jordan elimination method)

[A | I]

$$\left[\begin{array}{ccc|ccc} 5 & 2 & 1 & 1 & 0 & 0 \\ -1 & 3 & 0 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{W_1 \leftrightarrow W_2} \left[\begin{array}{ccc|ccc} -1 & 3 & 0 & 0 & 1 & 0 \\ 5 & 2 & 1 & 1 & 0 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{W_2 \to W_2 + 5W_1} \left[\begin{array}{ccc|ccc} -1 & 3 & 0 & 0 & 1 & 0 \\ 0 & 17 & 1 & 1 & 5 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{W_1 \to (-1)W_1} \left[\begin{array}{ccc|ccc} 1 & -3 & 0 & 0 & -1 & 0 \\ 0 & 17 & 1 & 1 & 5 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{W_2 \to W_2 - 8W_3}$$

$$\left[\begin{array}{ccc|ccc} 1 & -3 & 0 & 0 & -1 & 0 \\ 0 & 1 & 9 & 1 & 5 & -8 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[W_3 \to W_3 - 2W_2]{W_1 \to W_1 + 3W_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 27 & 3 & 14 & -24 \\ 0 & 1 & 9 & 1 & 5 & -8 \\ 0 & 0 & -19 & -2 & -10 & 17 \end{array}\right] \xrightarrow{W_3 \to \left(-\frac{1}{19}\right)W_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 27 & 3 & 14 & -24 \\ 0 & 1 & 9 & 1 & 5 & -8 \\ 0 & 0 & 1 & \frac{2}{19} & \frac{10}{19} & -\frac{17}{19} \end{array}\right] \xrightarrow[W_2 \to W_2 - 9W_3]{W_1 \to W_1 - 27W_3}$$

[I | A^{-1}]

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{3}{19} & -\frac{4}{19} & \frac{3}{19} \\ 0 & 1 & 0 & \frac{1}{19} & \frac{5}{19} & \frac{1}{19} \\ 0 & 0 & 1 & \frac{2}{19} & \frac{10}{19} & -\frac{17}{19} \end{array}\right]$$

$$\mathbf{A}^{-1} = \begin{bmatrix} \frac{3}{19} & \frac{-4}{19} & \frac{3}{19} \\ \frac{1}{19} & \frac{5}{19} & \frac{1}{19} \\ \frac{2}{19} & \frac{10}{19} & \frac{-17}{19} \end{bmatrix} = \frac{-1}{19} \begin{bmatrix} -3 & 4 & -3 \\ -1 & -5 & -1 \\ -2 & -10 & 17 \end{bmatrix}$$

Inverse matrix

Example 4

$$
\left[\mathbf{A}\middle|\mathbf{I}\right]=\left[\begin{array}{cccc|cccc}1&2&3&4&1&0&0&0\\-1&2&5&-2&0&1&0&0\\3&1&8&-4&0&0&1&0\\1&1&3&2&0&0&0&1\end{array}\right]\xrightarrow[W_4\to W_4-W_1]{\substack{W_2\to W_2+W_1\\W_3\to W_3-3W_1}}\left[\begin{array}{cccc|cccc}1&2&3&4&1&0&0&0\\0&4&8&2&1&1&0&0\\0&-5&-1&-16&-3&0&1&0\\0&-1&0&-2&-1&0&0&1\end{array}\right]\xrightarrow{W_2\leftrightarrow W_4}\left[\begin{array}{cccc|cccc}1&2&3&4&1&0&0&0\\0&-1&0&-2&-1&0&0&1\\0&-5&-1&-16&-3&0&1&0\\0&4&8&2&1&1&0&0\end{array}\right]\xrightarrow{W_2\to(-1)W_2}
$$

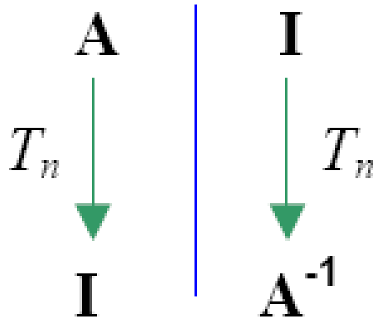
$$
\left[\begin{array}{cccc|cccc}1&2&3&4&1&0&0&0\\0&1&0&2&1&0&0&-1\\0&-5&-1&-16&-3&0&1&0\\0&4&8&2&1&1&0&0\end{array}\right]\xrightarrow[W_4\to W_4-4W_2]{\substack{W_1\to W_1-2W_2\\W_3\to W_3+5W_2}}\left[\begin{array}{cccc|cccc}1&0&3&0&-1&0&0&2\\0&1&0&2&1&0&0&-1\\0&0&-1&-6&2&0&1&-5\\0&0&8&-6&-3&1&0&4\end{array}\right]\xrightarrow[W_4\to W_4+8W_3]{W_1\to W_1+3W_3}\left[\begin{array}{cccc|cccc}1&0&0&-18&5&0&3&-13\\0&1&0&2&1&0&0&-1\\0&0&-1&-6&2&0&1&-5\\0&0&0&-54&13&1&8&-36\end{array}\right]
$$

$$
\xrightarrow[W_4\to\left(-\frac{1}{54}\right)W_4]{W_3\to(-1)W_3}\left[\begin{array}{cccc|cccc}1&0&0&-18&5&0&3&-13\\0&1&0&2&1&0&0&-1\\0&0&1&6&-2&0&-1&5\\0&0&0&1&-\frac{13}{54}&-\frac{1}{54}&-\frac{8}{54}&\frac{36}{54}\end{array}\right]\xrightarrow[W_3\to W_3-6W_4]{\substack{W_1\to W_1+18W_4\\W_2\to W_2-2W_4}}\left[\begin{array}{cccc|cccc}1&0&0&0&\frac{36}{54}&-\frac{18}{54}&\frac{18}{54}&-\frac{54}{54}\\0&1&0&0&\frac{80}{54}&\frac{2}{54}&\frac{16}{54}&-\frac{126}{54}\\0&0&1&0&-\frac{30}{54}&\frac{6}{54}&-\frac{6}{54}&\frac{54}{54}\\0&0&0&1&-\frac{13}{54}&-\frac{1}{54}&-\frac{8}{54}&\frac{36}{54}\end{array}\right]=\left[\mathbf{I}\middle|\mathbf{A}\right]
$$

Inverse matrix

Example 4

$$\mathbf{A}^{-1}=\begin{bmatrix}\frac{36}{54} & -\frac{18}{54} & \frac{18}{54} & -\frac{54}{54}\\ \frac{80}{54} & \frac{2}{54} & \frac{16}{54} & -\frac{126}{54}\\ -\frac{30}{54} & \frac{6}{54} & -\frac{6}{54} & \frac{54}{54}\\ -\frac{13}{54} & -\frac{1}{54} & -\frac{8}{54} & \frac{36}{54}\end{bmatrix}=\frac{1}{54}\begin{bmatrix}36 & -18 & 18 & -54\\ 80 & 2 & 16 & -126\\ -30 & 6 & -6 & 54\\ -13 & -1 & -8 & 36\end{bmatrix}$$



Verification :

$$\mathbf{A}^{-1}\mathbf{A}=\frac{1}{54}\begin{bmatrix}36 & -18 & 18 & -54\\ 80 & 2 & 16 & -126\\ -30 & 6 & -6 & 54\\ -13 & -1 & -8 & 36\end{bmatrix}\begin{bmatrix}1 & 2 & 3 & 4\\ -1 & 2 & 5 & -2\\ 3 & 1 & 8 & -4\\ 1 & 1 & 3 & 2\end{bmatrix}=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$